

# The Subelliptic Heat Kernel on the CR hyperbolic spaces

Jing Wang\*

Department of Mathematics, Purdue University  
West Lafayette, IN, USA

## Abstract

We study the heat kernel of the sub-Laplacian on the CR hyperbolic space  $\mathbb{H}^{2n+1}$  and on its universal covering  $\widetilde{\mathbb{H}^{2n+1}}$ . We work in cylindrical coordinates that reflects the symmetries coming from the fibration  $\mathbb{H}^{2n+1} \rightarrow \mathbb{CH}^n$ , and derive an explicit and geometrically meaningful formula for the subelliptic heat kernel. As a by-product we obtain the small-time asymptotics of the heat kernel, as well as an explicit formula for the sub-Riemannian distance.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>The sub-Laplacian on <math>\mathbb{H}^{2n+1}</math> and <math>\widetilde{\mathbb{H}^{2n+1}}</math></b>	<b>3</b>
2.1	Geometry of the standard complex hyperboloid . . . . .	3
2.2	The sub-Laplacian . . . . .	6
<b>3</b>	<b>The subelliptic heat kernel on <math>\mathbb{H}^{2n+1}</math> and <math>\widetilde{\mathbb{H}^{2n+1}}</math></b>	<b>11</b>
3.1	Integral representation of the heat kernel . . . . .	11
3.2	Asymptotics of the subelliptic heat kernel in small times . . . . .	14

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\*wang321@purdue.edu

# 1 Introduction

In this paper we study the subelliptic structure on the CR hyperbolic model  $\mathbb{H}^{2n+1}$  and on its universal covering  $\widetilde{\mathbb{H}^{2n+1}}$ . We will give the explicit integral representation of the subelliptic heat kernel and based on that, we derive the small time asymptotics of the kernel.

The key idea of this work is to observe that the sub-Riemannian structure of  $\mathbb{H}^{2n+1}$  comes from the fibration:

$$\mathbb{S}^1 \longrightarrow \mathbb{H}^{2n+1} \longrightarrow \mathbb{CH}^n$$

here  $\mathbb{H}^{2n+1}$  is the boundary of the unit ball in  $\mathbb{C}^{n+1}$  endowed with the hyperbolic metric, and  $\mathbb{CH}^n$  is the complex hyperbolic space of dimension  $n$ . In another word, we can consider  $\mathbb{H}^{2n+1}$  as a circle bundle over  $\mathbb{CH}^n$ . Its vertical space is then generated by the Reeb vector field, and its horizontal space is obtained by lifting the vector fields of  $\mathbb{CH}^n$ . Due to the fact that  $\mathbb{H}^{2n+1}$  is not simply connected, we will also study its universal covering  $\widetilde{\mathbb{H}^{2n+1}}$ , which possesses the sub-Riemannian structure of a line bundle over  $\mathbb{CH}^n$ .

$\widetilde{\mathbb{H}^{2n+1}}$  is a model space of CR-manifold with vanishing pseudo-Hermitian torsion, that is, a Sasakian manifold with constant negative sectional curvature (see[11]). This special geometric structure implies the commutation between the Reeb vector field  $T$  of the CR structure of  $\widetilde{\mathbb{H}^{2n+1}}$  and the sub-Laplacian  $L$ . It then allows us to write the integral representation of the subelliptic heat kernel on  $\widetilde{\mathbb{H}^{2n+1}}$ . By applying the steepest descent method, we are able to derive the small-time asymptotics of the subelliptic heat kernel as well as the explicit formula for the sub-Riemannian distance. The small-time asymptotics has three different behaviors on  $\widetilde{\mathbb{H}^{2n+1}}$ : on diagonal points, on the cut-locus, and off the cut-locus.

Beals-Greiner-Stanton first studied the small-time asymptotics of subelliptic heat kernel on CR manifolds by using pseudodifferential calculus (see [6]). After that, the study of the heat kernel on the Heisenberg group, as a model of flat Sasakian manifold, began with Beals-Gaveau-Greiner (see [12]). In [12], he first established the integral representation of the subelliptic heat kernel, which plays a crucial role in obtaining the asymptotics of the heat kernel in small time. Later on, Baudoin and Bonnefont (see [3]) studied the heat kernel on the 3-dimensional sub-Riemannian manifold with positive constant sectional curvature, which may also be identified with the group  $SU(2)$ . As a multidimensional extension, the study of the heat kernel on the CR sphere  $\mathbb{S}^{2n+1}$  was done in [4]. In the case of negative curvature, Bonnefont gave similar results (see [8]) on the group  $SL(2, \mathbb{R})$ , as the model space of a negatively curved 3-dimensional sub-Riemannian Sasakian manifold. As a generalization, in this paper, we study the heat kernel of the  $n$ -dimensional Sasakian manifold with negative constant sectional curvature.

The study of finding explicit formulas for subelliptic heat kernels has generated a great amount of work (see [1], [2], [3], [4], [5] [8], [12] and the references therein). Explicit expressions have numerous applications, for instance: determination of sharp constant in functional inequalities (see [7], [15]), computation of the sub-Riemannian metric (see

[3]), sharp upper and lower bounds for the heat kernel (see [5],[10]), and semigroup sub-commutations (see [14]). However, despite such numerous works, few explicit and tractable formulas are actually known and most of them are restricted to a Lie group framework. As a counterpart to the models in [12] and [4], the present work gives explicit and tractable expressions that hold in a natural sub-Riemannian model with negative sectional curvature.

The paper is organized as follows. In the first section we study the geometry of the complex hyperboloid in  $\mathbb{C}^{n+1}$ . We show that it is naturally endowed with a CR structure which is conformally equivalent to the CR structure of the Heisenberg group. We introduce geometrically meaningful coordinates that reflect the symmetries of the natural fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{H}^{2n+1} \longrightarrow \mathbb{C}\mathbb{H}^n.$$

We then study the sub-Laplacian in these coordinates and show in particular that the radial part of it can be written

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + ((2n-1) \coth r + \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial \theta^2}.$$

Based on this expression we deduce our formulas for the subelliptic heat kernels on  $\mathbb{H}^{2n+1}$  and  $\widetilde{\mathbb{H}^{2n+1}}$ . The expressions we obtain are precise enough to deduce from them the small-time asymptotics of the kernel. As a consequence, we obtain an expression for the sub-Riemannian distance.

## 2 The sub-Laplacian on $\mathbb{H}^{2n+1}$ and $\widetilde{\mathbb{H}^{2n+1}}$

### 2.1 Geometry of the standard complex hyperboloid

We consider the odd dimensional hyperbolic space  $\mathbb{H}^{2n+1}$ , which may be identified with the unit ball of  $\mathbb{C}^{n+1}$  for the Minkowski metric:

$$\|z\|_H^2 = |z_{n+1}|^2 - \sum_{k=1}^n |z_k|^2,$$

that is

$$\mathbb{H}^{2n+1} = \{z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1}, \|z\|_H = 1\}.$$

Let us first observe that  $\mathbb{H}^{2n+1}$  appears as the boundary of a domain which is biholomorphically equivalent to the Siegel domain. Let  $\mathbf{B}^{n+1} = \{z \in \mathbb{C}^{n+1} : \|z\|_H < 1\}$  be the unit ball in  $\mathbb{C}^{n+1}$ .

Consider the Cayley transform  $\mathbf{B}^{n+1} \longrightarrow \Omega_{n+1}^c$ ,

$$\mathcal{C}(z) = \left( \frac{z_1}{z_{n+1} - 1}, \dots, \frac{z_n}{z_{n+1} - 1}, i \frac{z_{n+1} + 1}{z_{n+1} - 1} \right)$$

where

$$\Omega_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \mathbf{Im}(w) > \|z\|^2\},$$

is the Siegel domain and  $\|\cdot\|$  is the standard norm in  $\mathbb{C}^n$ ,  $\mathcal{C}$  is a biholomorphism.

In particular, the restriction on to the surface minus a point, gives a CR diffeomorphism on to the boundary of the Siegel domain.

$$\mathcal{C} : \mathbb{H}^{2n+1} \setminus \{e_{n+1}\} \longrightarrow \partial\Omega_{n+1}$$

where  $e_{n+1}$  is the north pole <sup>1</sup>.

Due to the strict pseudo-convexticity of  $\partial\Omega_{n+1}$  (see [9]), we can conclude the strict pseudo-convexticity of  $\mathbb{H}^{2n+1}$ .

Moreover, we have a well known CR diffeomorphism from the Heisenberg group  $\mathbf{H}_n$  to the boundary of the Siegel domain

$$f : \mathbf{H}_n \longrightarrow \partial\Omega_{n+1}, \quad f(z, t) = (z, t + i|z|^2),$$

with inverse

$$f^{-1}(z, w) = (z, \mathbf{Re}(w)), \quad z \in \mathbb{C}^n, \quad w \in \mathbb{C}$$

Thus we obtain the CR equivalence between the hyperbolic space and the Heisenberg group which is given by:

$$F : \mathbb{H}^{2n+1} \setminus \{e_{n+1}\} \longrightarrow \mathbf{H}_n, \quad F = f^{-1} \circ \mathcal{C}$$

It is now time to introduce the CR structure on  $\mathbb{H}^{2n+1}$ . There is a canonical contact form  $\eta_0$  on  $\mathbf{H}_n$  given as follows (see [9]),

$$\eta_0 = dt + i \sum_{j=1}^n (z_j d\bar{z}_j - \bar{z}_j dz_j).$$

It is a pseudo-Hermitian structure on  $(\mathbf{H}_n, T_{1,0}(\mathbf{H}_n))$  which shows the strict pseudo-convextity. Here  $T_{1,0}(\mathbf{H}_n)$  denotes the CR structure on  $\mathbf{H}_n$ . By pulling back  $\eta_0$ , we obtain a contact form  $\eta_1$  on  $\mathbb{H}^{2n+1}$ , i.e.

$$\eta_1 = F^* \eta_0 = \frac{i}{|z_{n+1} - 1|^2} \left( \sum_{j=1}^n (z_j d\bar{z}_j - \bar{z}_j dz_j) - (z_{n+1} d\bar{z}_{n+1} - \bar{z}_{n+1} dz_{n+1}) \right).$$

The associated Levi form  $-id\eta_1$  then defines a sub-Riemannian structure on tangent space of  $\mathbb{H}^{2n+1}$ .

There is a natural group action of  $\mathbb{S}^1$  on  $\mathbb{H}^{2n+1}$  which is defined by

$$(z_1, \dots, z_n) \rightarrow (e^{i\theta} z_1, \dots, e^{i\theta} z_n).$$

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<sup>1</sup> We call north pole the point with complex coordinates  $z_1 = 0, \dots, z_{n+1} = 1$ , it is therefore the point with real coordinates  $(0, \dots, 0, 1, 0)$ .

The generator of this action shall be denoted by  $T$  throughout the paper. We have for all  $f \in C^\infty(\mathbb{H}^{2n+1})$

$$Tf(z) = \frac{d}{d\theta} f(e^{i\theta} z) |_{\theta=0},$$

so that

$$T = i \sum_{j=1}^{n+1} \left( z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

This action induces a fibration (circle bundle) from  $\mathbb{H}^{2n+1}$  to the projective complex space  $\mathbb{CP}^n$ .

Let  $\eta$  be the standard pseudo-Hermitian contact form on  $\mathbb{H}^{2n+1}$ , then

$$\eta = \frac{i}{2} \left( \sum_{j=1}^n (\bar{z}_j dz_j - z_j d\bar{z}_j) - (\bar{z}_{n+1} dz_{n+1} - z_{n+1} d\bar{z}_{n+1}) \right). \quad (2.1)$$

The vector field  $T$  is the Reeb vector field (characteristic direction) of  $\eta$ , i.e.,  $\eta(T) = 1$ , and we can easily find that  $\eta = \frac{|z_{n+1}-1|^2}{2} \eta_1$ .

**Remark 2.1** *In the case of  $n = 1$ , we have*

$$\mathbb{H}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2, |z_2|^2 - |z_1|^2 = 1\}. \quad (2.2)$$

*It is isomorphic to the group  $SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$ .*

*Indeed, by writing  $z_1 = x_1 + iy_1$ ,  $z_2 = x_2 + iy_2$ ,  $x_1, x_2, y_1, y_2 \in \mathbb{R}$  we see that (2.2) becomes*

$$x_2^2 + y_2^2 - x_1^2 - y_1^2 = 1$$

*that is*

$$(x_1 + x_2)(x_1 - x_2) - (y_1 + y_2)(y_1 - y_2) = 1$$

*Then by denoting*

$$a = x_1 + x_2, d = x_1 - x_2, b = y_1 + y_2, c = y_1 - y_2,$$

*we can obtain the isomorphism between the two spaces.*

*In this case  $\mathbb{H}^3$  has a special group structure, and the study of the sub-Laplacian and the corresponding heat kernel is done in [8].*

Notice that  $\mathbb{H}^{2n+1}$  is not simply connected. In the sequel we will also study the universal covering of  $\mathbb{H}^{2n+1}$  which shall be denoted by  $\widetilde{\mathbb{H}^{2n+1}}$ .

## 2.2 The sub-Laplacian

To study the sub-Laplacian, let us first focus on the  $\mathbb{H}^{2n+1}$  case. Then we will lift the subelliptic structure to  $\widetilde{\mathbb{H}^{2n+1}}$ .

We now introduce a set of coordinates that takes into account the symmetries of the fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{H}^{2n+1} \longrightarrow \mathbb{CH}^n$$

where  $\mathbb{CH}^n$  is the complex hyperbolic space that is the unit ball in  $\mathbb{C}^n$ :

$$\{v = (v_1, \dots, v_n) \in \mathbb{C}^n, \|v\| < 1\}$$

endowed with the metric

$$\Omega = \sum_{j,k=1}^n u_j \bar{v}_k dv_j \wedge d\bar{v}_k \quad (2.3)$$

where  $\|v\|^2 = \sum_{k=1}^n |v_k|^2$ ,  $u = -\log(1 - (\sum_{k=1}^n v_k^2))$ , and  $u_k = \frac{\partial u}{\partial v_k}$ , for  $k = 1, \dots, n$ .

The new coordinates are given as follows.

Let  $(w_1, \dots, w_n, \theta)$  be local coordinates for  $\mathbb{H}^{2n+1}$ , where  $(w_1, \dots, w_n)$  are the local inhomogeneous coordinates for  $\mathbb{CH}^n$  given by  $w_j = z_j/z_{n+1}$ , and  $\theta$  is the local fiber coordinate.

That is,  $(w_1, \dots, w_n)$  parametrizes the complex lines passing through the north pole, while  $\theta$  determines a point on the line that is of unit distance from the north pole. More explicitly, these coordinates are given by

$$(w_1, \dots, w_n, \theta) \longrightarrow \left( \frac{w_1 e^{i\theta}}{\sqrt{1-\rho^2}}, \dots, \frac{w_n e^{i\theta}}{\sqrt{1-\rho^2}}, \frac{e^{i\theta}}{\sqrt{1-\rho^2}} \right), \quad (2.4)$$

where  $\rho = \sqrt{\sum_{j=1}^n |w_j|^2}$ ,  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ , and  $w \in \mathbb{CH}^n$ . In these coordinates, it is clear that

$$T = \frac{\partial}{\partial \theta}. \quad (2.5)$$

By using the diffeomorphism

$$(w_1, \dots, w_n, \theta, \kappa) \longrightarrow \left( \frac{\kappa w_1 e^{i\theta}}{\sqrt{1-\rho^2}}, \dots, \frac{\kappa w_n e^{i\theta}}{\sqrt{1-\rho^2}}, \frac{\kappa e^{i\theta}}{\sqrt{1-\rho^2}} \right),$$

and then restrict to the surface on which we have:

$$\kappa = 1, \frac{\partial}{\partial \kappa} = 0,$$

we compute that on  $\mathbb{H}^{2n+1}$ , for  $1 \leq k \leq n$

$$\begin{aligned} \frac{\partial}{\partial z_k} &= \sqrt{1-\rho^2} e^{-i\theta} \frac{\partial}{\partial w_k} \\ \frac{\partial}{\partial \bar{z}_k} &= \sqrt{1-\rho^2} e^{i\theta} \frac{\partial}{\partial \bar{w}_k} \end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial z_{n+1}} &= -\sqrt{1-\rho^2}e^{-i\theta} \left( \sum_{j=1}^n w_j \frac{\partial}{\partial w_j} - \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \\ \frac{\partial}{\partial \bar{z}_{n+1}} &= -\sqrt{1-\rho^2}e^{i\theta} \left( \sum_{j=1}^n \bar{w}_j \frac{\partial}{\partial \bar{w}_j} + \frac{1}{2i} \frac{\partial}{\partial \theta} \right).\end{aligned}$$

Plug into (2.1), we obtain

$$\eta = -\frac{i}{2(1-\rho^2)} \sum_{j=1}^n (w_j d\bar{w}_j - \bar{w}_j dw_j) + d\theta.$$

The Levi form is then given by

$$L_\eta = id\eta = \frac{1}{1-\rho^2} \sum_{j=1}^n dw_j \wedge d\bar{w}_j + \frac{1}{(1-\rho^2)^2} \sum_{j,k=1}^n \bar{w}_k w_j dw_k \wedge d\bar{w}_j, \quad (2.6)$$

which coincides with the metric  $\Omega$  on  $\mathbb{CH}^n$ . (see (2.3))

Let  $H(\mathbb{H}^{2n+1})$  denote the Levi distribution of  $T(\mathbb{H}^{2n+1})$ , on which we define the bilinear form  $G_\eta$  (as well as on its  $\mathbb{C}$ -bilinear extension) as follows

$$G_\eta = \frac{1}{2} L_\eta.$$

The Webster metric then may be given as

$$g_\eta = \pi_H G_\eta + \eta \otimes \eta,$$

where  $\pi_H : T(\mathbb{H}^{2n+1}) \rightarrow H(\mathbb{H}^{2n+1})$  is the projection associated with the direct sum decomposition

$$T(\mathbb{H}^{2n+1}) = H(\mathbb{H}^{2n+1}) \oplus \mathbb{R}T$$

and  $(\pi_H G_\eta)(X, Y) = G_\eta(\pi_H X, \pi_H Y)$ , for any  $X, Y \in T(\mathbb{H}^{2n+1})$ .

Now consider the horizontal lift of the vector fields  $\{\frac{\partial}{\partial w_k}\}_{k=1}^n$  on  $T(\mathbb{CH}^n)$ , and denote them as  $\{T_k\}_{k=1}^n$ , we have

$$T_k = \sqrt{1-\rho^2}e^{-i\theta} \left( \frac{\partial}{\partial w_k} + \frac{\bar{w}_k}{2i(1-\rho^2)} \frac{\partial}{\partial \theta} \right), \quad k = 1, \dots, n \quad (2.7)$$

Observe that  $\{T_k, \bar{T}_k\}_{k=1}^n$  forms a basis of the horizontal space of  $T(\mathbb{H}^{2n+1})$ . Our goal is now to compute the sub-Laplacian  $L$ .

Let us recall that on a strict pseudoconvex CR manifold  $M$  endowed with a pseudo-Hermitian structure  $\eta$ , the sub-Laplacian  $L$  is given by (see [9])

$$Lu = \mathbf{trace}_{G_\eta} \{ \pi_H \nabla^2 u \}$$

for any  $u \in C^2(\mathbb{H}^{2n+1})$ . Now we deduce the following expression of  $L$  on  $\mathbb{H}^{2n+1}$ .

**Proposition 2.2** *Let  $L$  be the sub-Laplacian on  $\mathbb{H}^{2n+1}$  with respect to the contact form  $\eta$ . In the coordinates (2.4), we have*

$$L = 4(1 - \rho^2) \sum_{k=1}^n \frac{\partial^2}{\partial w_k \partial \overline{w_k}} - 4(1 - \rho^2) \mathcal{R} \overline{\mathcal{R}} + \rho^2 \frac{\partial^2}{\partial \theta^2} + 2i(1 - \rho^2)(\mathcal{R} - \overline{\mathcal{R}}) \frac{\partial}{\partial \theta}.$$

where  $\mathcal{R} = \sum_{k=1}^n w_k \frac{\partial}{\partial w_k}$ .

*Proof.* First notice that for any local frame  $\{T_k\}_{k=1}^n$  of  $T_{1,0}(\mathbb{H}^{2n+1})$ , let  $h_{k\bar{j}} = L_\eta(T_k, T_{\bar{j}})$ , we have

$$Lu = \sum_{k,j=1}^n \left( h^{\bar{k}j} (\nabla^2 u)(T_k, \overline{T_j}) + h^{k\bar{j}} (\nabla^2 u)(\overline{T_k}, T_j) \right), \quad (2.8)$$

where  $[h^{k\bar{j}}] = [h_{k\bar{j}}]^{-1}$  (see [9], P117). By (2.6), we compute that

$$h^{k\bar{j}} = 2(\delta_{kj} - \overline{w_k} w_j), \quad k, j = 1, \dots, n \quad (2.9)$$

Plug (2.7) and (2.9) into (2.8) we obtain

$$L = 4(1 - \rho^2) \sum_{k=1}^n \frac{\partial^2}{\partial w_k \partial \overline{w_k}} - 4(1 - \rho^2) \mathcal{R} \overline{\mathcal{R}} + \rho^2 \frac{\partial^2}{\partial \theta^2} + 2i(1 - \rho^2)(\mathcal{R} - \overline{\mathcal{R}}) \frac{\partial}{\partial \theta}.$$

□

From (2.6) we know that the volume form  $\Psi$  on  $\mathbb{H}^{2n+1}$  is given by

$$\Psi = \eta \wedge (d\eta)^n = \frac{i^{n^2} n!}{(1 - \rho^2)^{2n}} \det(\alpha_{k\bar{j}}) d\theta \wedge dw_1 \wedge \dots \wedge dw_n \wedge d\overline{w_1} \wedge \dots \wedge d\overline{w_n}$$

where  $\alpha_{k\bar{k}} = 1, \alpha_{k\bar{j}} = \overline{w_k} w_j$ , for  $1 \leq k, j \leq n, k \neq j$ . We can observe that  $L$  is symmetric with respect to this measure. Since  $\mathbb{H}^{2n+1}$  is complete, we deduce that  $L$  is essentially self-adjoint on  $C_0^\infty(\mathbb{H}^{2n+1})$ .

**Remark 2.3** *The sub-Laplacian turns out to be related to a natural hyperbolic operator on  $\mathbb{H}^{2n+1}$ . Indeed, let  $\tilde{\square}$  denote the hyperbolic operator on  $\mathbb{H}^{2n+1}$ , we can describe it from the one on  $\mathbb{C}^{n+1}$ . The hyperbolic operator on  $\mathbb{C}^{n+1}$  writes*

$$\square_{\mathbb{C}} = 4 \left( \sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \overline{z_k}} - \frac{\partial^2}{\partial z_{n+1} \partial \overline{z_{n+1}}} \right).$$

Let  $f : \mathbb{H}^{2n+1} \rightarrow \mathbb{C}$  be a smooth function, then it may be extended to  $\mathbb{C}^{n+1} \setminus \{0\}$  by

$$\tilde{f}(z) = f\left(\frac{z}{\|z\|}\right),$$



then we have on  $\mathbb{H}^{2n+1}$

$$\square f(z) = \square_{\mathbb{C}} \tilde{f}(z).$$

Simple but tedious calculations then show that

$$\square = 4 \left( \sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \bar{z}_k} - \frac{\partial^2}{\partial z_{n+1} \partial \bar{z}_{n+1}} \right) + (\mathcal{S} + \bar{\mathcal{S}})^2 + (2n+1)(\mathcal{S} + \bar{\mathcal{S}}).$$

where  $\mathcal{S} = \sum_{k=1}^n z_k \frac{\partial}{\partial z_k}$ . In the coordinates (2.4) it writes:

$$\square = 4(1 - \rho^2) \sum_{k=1}^n \frac{\partial^2}{\partial w_k \partial \bar{w}_k} - 4(1 - \rho^2) \mathcal{R} \bar{\mathcal{R}} - (1 - \rho^2) \frac{\partial^2}{\partial \theta^2} + 2i(1 - \rho^2)(\mathcal{R} - \bar{\mathcal{R}}) \frac{\partial}{\partial \theta}.$$

From this we can observe that  $L$  is related to  $\square$  by the formula:

$$L = \square + T^2, \quad (2.10)$$

here  $T = \frac{\partial}{\partial \theta}$ .

**Remark 2.4** We can observe, a fact which will be important for us, that  $L$  and  $T$  commute, that is, on smooth functions

$$TL = LT.$$

Due to the symmetries of the fibration  $\mathbb{H}^{2n+1} \rightarrow \mathbb{C}\mathbb{H}^n$ , in the study of the heat kernel, it will be enough to compute the radial part of  $L$  with respect to the variables  $(\rho, \theta)$ .

Let us consider the following second order differential operator

$$\tilde{L} = (1 - \rho^2)^2 \frac{\partial^2}{\partial \rho^2} + (1 - \rho^2) \left( \frac{(2n-1)}{\rho} - \rho \right) \frac{\partial}{\partial \rho} + \rho^2 \frac{\partial^2}{\partial \theta^2},$$

which is defined on the space  $\mathcal{D}$  of smooth functions  $f : \mathbb{R}_{\geq 0} \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}$  that satisfies  $\frac{\partial f}{\partial \rho} = 0$  if  $\rho = 0$ . It is seen that  $\tilde{L}$  is essentially self-adjoint on  $\mathcal{D}$  with respect to the measure  $\frac{\rho^{2n-1}}{(1-\rho^2)^{n+1}} d\rho d\theta$ .

**Proposition 2.5** Let us denote by  $\psi$  the map from  $\mathbb{H}^{2n+1}$  to  $\mathbb{R}_{\geq 0} \times \mathbb{R}/2\pi\mathbb{Z}$  such that

$$\psi \left( \frac{w_1 e^{i\theta}}{\sqrt{1 - \rho^2}}, \dots, \frac{w_n e^{i\theta}}{\sqrt{1 - \rho^2}}, \frac{e^{i\theta}}{\sqrt{1 - \rho^2}} \right) = (\rho, \theta).$$

For every  $f \in \mathcal{D}$ , we have

$$L(f \circ \psi) = (\tilde{L}f) \circ \psi.$$

*Proof.* Notice that by symmetries, we have

$$\left( \sum_{k=1}^n \frac{\partial^2}{\partial w_k \partial \bar{w}_k} \right) (f \circ \psi) = \left( \left( \frac{1}{4} \frac{\partial^2}{\partial \rho^2} + \frac{2n-1}{4\rho} \frac{\partial}{\partial \rho} \right) f \right) \circ \psi$$

and

$$\mathcal{R}(f \circ \psi) = \overline{\mathcal{R}}(f \circ \psi) = \left( \left( \frac{1}{2} \rho \frac{\partial}{\partial \rho} \right) f \right) \circ \psi.$$

Together with Proposition 2.2, we have the conclusion.  $\square$

Finally, instead of  $\rho$ , it will be expedient to introduce the variable  $r$  which is defined by  $\rho = \tanh r$ . It is then easy to see that we can write  $\tilde{L}$  as

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + ((2n-1) \coth r + \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial \theta^2}. \quad (2.11)$$

This is the expression of  $\tilde{L}$  which shall be the most convenient for us and that is going to be used throughout the paper.

The invariant and symmetric measure for  $\tilde{L}$  is given by

$$d\mu_r = \frac{2\pi^n}{\Gamma(n)} (\sinh r)^{2n-1} \cosh r dr d\theta.$$

**Remark 2.6** In the case of  $\mathbb{H}^3$  ( $n = 1$ ), which is isomorphic to the Lie group  $SL(2)$ , we obtain

$$\begin{aligned} \tilde{L} &= (1 - \rho^2)^2 \frac{\partial^2}{\partial \rho^2} + \left( \frac{(1 - \rho^2)^2}{\rho} \right) \frac{\partial}{\partial \rho} + \rho^2 \frac{\partial^2}{\partial \theta^2} \\ &= \frac{\partial^2}{\partial r^2} + 2 \coth 2r \frac{\partial}{\partial r} + \tanh^2 r \frac{\partial^2}{\partial \theta^2} \end{aligned}$$

This coincides with the result in [8].

**Remark 2.7** By Remark 2.3, we see that the radial part of the hyperbolic operator in cylindrical coordinates is

$$\tilde{\square} = \frac{\partial^2}{\partial r^2} + ((2n-1) \coth r + \tanh r) \frac{\partial}{\partial r} - \frac{1}{\cosh^2 r} \frac{\partial^2}{\partial \theta^2}. \quad (2.12)$$

On the other hand, since in the coordinates (2.4), we have  $z_{n+1} = \cosh r e^{i\theta}$ , it is clear that the Riemannian distance  $\delta$  from the north pole satisfies

$$\cosh \delta = \cosh r \cos \theta.$$

An easy calculation shows that by making the change of variable  $\cosh \delta = \cosh r \cos \theta$ , the operator  $\tilde{\square}$  acts on the functions depending only on  $\delta$  as

$$\frac{\partial^2}{\partial \delta^2} + 2n \coth \delta \frac{\partial}{\partial \delta}.$$

We denote it as  $\tilde{\Delta}$ , which is known to be the expression of the radial part of Laplace-Beltrami operator on hyperbolic spaces in spherical coordinates, with symmetric measure  $\frac{2\pi^n}{\Gamma(n)}(\sinh \delta)^{2n} d\delta$ .

Let us now move to the case of the universal covering  $\widetilde{\mathbb{H}^{2n+1}}$ . First note that  $\mathbb{H}^{2n+1}$  is the complex hyperbolic space  $\mathbb{CH}^n$  endowed with a circle bundle  $\mathbb{S}^1$  structure, and therefore,  $\widetilde{\mathbb{H}^{2n+1}}$  is  $\mathbb{CH}^n$  endowed with a line bundle  $\mathbb{R}$  structure. In the local coordinates, it is represented as

$$(w_1, \dots, w_n, \theta) \in \mathbb{C}^n \times \mathbb{R}$$

and the projection from  $\widetilde{\mathbb{H}^{2n+1}}$  to  $\mathbb{H}^{2n+1}$  is obtained by projecting the  $\theta$  part on  $\mathbb{R}/2\pi\mathbb{Z}$ . Clearly, the lift of the vector fields  $\{T_k\}_{k=1}^n$ ,  $T$  are still given by the same expression as in (2.7) and (2.5), but are now defined for all  $(w_1, \dots, w_n, \theta) \in \mathbb{C}^n \times \mathbb{R}$ .

The subelliptic operator  $L$  (as well as its radial part  $\tilde{L}$ ) defined on  $\widetilde{\mathbb{H}^{2n+1}}$  obviously shares the same properties as the ones on  $\mathbb{H}^{2n+1}$ .

### 3 The subelliptic heat kernel on $\mathbb{H}^{2n+1}$ and $\widetilde{\mathbb{H}^{2n+1}}$

#### 3.1 Integral representation of the heat kernel

It is known (see [13]) that the Riemannian heat kernel associated to  $\tilde{\Delta}$  issued from the north pole writes

$$q_t(\cosh \delta) = \frac{\Gamma(n+1)e^{-n^2 t}}{(2\pi)^{n+1}\sqrt{\pi t}} \int_0^{+\infty} \frac{e^{\frac{\pi^2 - u^2}{4t}} \sinh u \sin \frac{\pi u}{2t}}{(\cosh u + \cosh \delta)^{n+1}} du \quad (3.13)$$

where, as above,  $\delta$  is the Riemannian distance from the north pole. Another useful formula for the heat kernel  $q_t$  which shall be used later in obtaining the small-time asymptotics is as follows: (see [16], P105)

$$q_t(\cosh \delta) = \frac{e^{-n^2 t}}{\sqrt{4\pi t}} \left( -\frac{1}{2\pi \sinh \delta} \frac{\partial}{\partial \delta} \right)^n \left( e^{-\frac{\delta^2}{4t}} \right). \quad (3.14)$$

Easy calculation shows that  $q_t$  satisfies the following equations:

$$\frac{\partial}{\partial t} q_t(\cosh r \cos \theta) = \tilde{\square} q_t(\cosh r \cos \theta)$$

where  $\tilde{\square}$  as defined in (2.12) is a hyperbolic operator on  $(0, \infty) \times [-\pi, \pi]$ , which is symmetric with respect to the measure  $\frac{2\pi^n}{\Gamma(n)}(\sinh r)^{2n-1} \cosh r dr d\theta$ . We also have

$$\frac{\partial}{\partial t} q_t(\cosh r \cosh y) = \tilde{\Delta} q_t(\cosh r \cosh y) \quad (3.15)$$

where

$$\tilde{\Delta} = \frac{\partial^2}{\partial r^2} + ((2n-1) \coth r + \tanh r) \frac{\partial}{\partial r} + \frac{1}{\cosh^2 r} \frac{\partial^2}{\partial y^2}.$$

is a elliptic operator defined  $(0, +\infty) \times \mathbb{R}$ , it is essentially self-adjoint with respect to the measure  $\frac{2\pi^n}{\Gamma(n)} (\sinh r)^{2n-1} \cosh r dr dy$ .

The key idea is to observe that since  $\square$  and  $\frac{\partial}{\partial \theta}$  commute, by (2.10) we formally have

$$e^{tL} = e^{t \frac{\partial^2}{\partial \theta^2}} e^{t \square}. \quad (3.16)$$

This gives a way to express the sub-Riemannian heat kernel in terms of the Riemannian one. Using the commutation (3.16) and the formula  $\cosh \delta = \cosh r \cosh \theta$ , we then infer the following proposition.

**Proposition 3.1** *For  $t > 0$ ,  $r \in [0, +\infty)$ ,  $\theta \in (-\infty, +\infty)$ , the subelliptic heat kernel on  $\widetilde{\mathbb{H}^{2n+1}}$  is given by*

$$p_t^{\widetilde{\mathbb{H}^{2n+1}}}(r, \theta) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-i\theta)^2}{4t}} q_t(\cosh r \cosh y) dy, \quad (3.17)$$

more precisely,

$$p_t^{\widetilde{\mathbb{H}^{2n+1}}}(r, \theta) = \frac{\Gamma(n+1) e^{-n^2 t + \frac{\pi^2}{4t}}}{(2\pi)^{n+2} t} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{e^{\frac{(y-i\theta)^2 - u^2}{4t}} \sinh u \sin\left(\frac{\pi u}{2t}\right)}{(\cosh u + \cosh r \cosh y)^{n+1}} du dy. \quad (3.18)$$

*Proof.* Let

$$h_t(r, \theta) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-i\theta)^2}{4t}} q_t(\cosh r \cosh y) dy,$$

and  $L_0 = \frac{\partial^2}{\partial r^2} + ((2n-1) \coth r + \tanh r) \frac{\partial}{\partial r}$ . Using the fact that

$$\frac{\partial}{\partial t} \left( \frac{e^{\frac{(y-i\theta)^2}{4t}}}{\sqrt{4\pi t}} \right) = \frac{\partial^2}{\partial \theta^2} \left( \frac{e^{\frac{(y-i\theta)^2}{4t}}}{\sqrt{4\pi t}} \right) = -\frac{\partial^2}{\partial y^2} \left( \frac{e^{\frac{(y-i\theta)^2}{4t}}}{\sqrt{4\pi t}} \right), \quad (3.19)$$

and by (3.15) we have

$$\frac{\partial}{\partial t} (q_t(\cosh r \cosh y)) = \left( L_0 + \frac{1}{\cosh^2 r} \frac{\partial^2}{\partial y^2} \right) (q_t(\cosh r \cosh y)).$$

A double integration by parts with respect to  $y$  gives us that

$$\frac{\partial}{\partial t} (h_t(r, \theta)) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-i\theta)^2}{4t}} \left( L_0 - \tanh^2 r \frac{\partial^2}{\partial y^2} \right) q_t(\cosh r \cosh y) dy$$

On the other hand, since  $\tilde{L} = L_0 + \tanh^2 r \frac{\partial^2}{\partial \theta^2}$ ,

$$\tilde{L}(h_t(r, \theta)) = \int_{-\infty}^{+\infty} \left( \left( \tanh^2 r \frac{\partial^2}{\partial \theta^2} \left( \frac{e^{\frac{(y-i\theta)^2}{4t}}}{\sqrt{4\pi t}} \right) \right) q_t(\cosh r \cosh y) + \left( \frac{e^{\frac{(y-i\theta)^2}{4t}}}{\sqrt{4\pi t}} \right) L_0(q_t(\cosh r \cosh y)) \right) dy$$

By (3.19) and another double integration by parts with respect to  $y$  we obtain that

$$\tilde{L}(h_t(r, \theta)) = \frac{\partial}{\partial t} (h_t(r, \theta))$$

On the other hand, it suffices to check the initial condition for functions of the form  $f(r, \theta) = e^{i\lambda\theta} g(r)$  where  $\lambda \in \mathbb{R}$  and  $g$  is smooth. We observe that

$$\begin{aligned} h_t * f &= \int_{r>0} \int_{\theta=-\infty}^{+\infty} h_t(r, \theta) f(r, \theta) d\mu_r \\ &= \frac{2\pi^n}{\Gamma(n)} \int_{r>0} \int_{\theta=-\infty}^{+\infty} \int_{y>0} \left( \frac{e^{-\frac{(\theta+iy)^2}{4t}} + e^{-\frac{(\theta-iy)^2}{4t}}}{\sqrt{4\pi t}} \right) q_t(\cosh r \cosh y) e^{i\lambda\theta} g(r) (\sinh r)^{2n-1} \cosh r dy d\theta dr \end{aligned}$$

Now by changing the contour of the integral we get:

$$\int_{\theta=-\infty}^{+\infty} \left( \frac{e^{-\frac{(\theta-iy)^2}{4t}}}{\sqrt{4\pi t}} \right) e^{i\lambda\theta} d\theta = e^{\lambda y} \int_{\theta=-\infty}^{+\infty} \left( \frac{e^{-\frac{\theta^2}{4t}}}{\sqrt{4\pi t}} \right) e^{i\lambda\theta} d\theta = e^{\lambda y - \lambda^2 t}$$

and

$$\int_{\theta=-\infty}^{+\infty} \left( \frac{e^{-\frac{(\theta+iy)^2}{4t}}}{\sqrt{4\pi t}} \right) e^{i\lambda\theta} d\theta = e^{-\lambda y} \int_{\theta=-\infty}^{+\infty} \left( \frac{e^{-\frac{\theta^2}{4t}}}{\sqrt{4\pi t}} \right) e^{i\lambda\theta} d\theta = e^{-\lambda y - \lambda^2 t}$$

Hence

$$\begin{aligned} h_t * f &= \int_{r>0} \int_{\theta=-\infty}^{+\infty} h_t(r, \theta) f(r, \theta) d\mu_r \\ &= \frac{4\pi^n}{\Gamma(n)} e^{-\lambda^2 t} \int_{r>0} \int_{y>0} q_t(\cosh r \cosh y) g(r) \cosh(\lambda y) (\sinh r)^{2n-1} \cosh r dy dr \\ &= \int_{r>0} \int_{y=-\infty}^{+\infty} q_t(\cosh r \cosh y) l(r, y) d\mu_r \\ &= e^{-\lambda^2 t} e^{t\tilde{\Delta}}(l)(0) \end{aligned}$$

where  $l(r, y) = g(r) \cosh(\lambda y)$ , and it tends to  $l(0)$  as  $t$  goes to 0. Therefore  $h_t * f$  converges to 0 as  $t$  goes to 0.

Thus  $h_t(r, \theta)$  is the desired subelliptic heat kernel. We then obtain the expression  $\widehat{p_t^{\mathbb{H}^{2n+1}}}(r, \theta)$  by plugging in (3.13).  $\square$

We can then easily deduce the subelliptic heat kernel on  $\mathbb{H}^{2n+1}$ .

**Proposition 3.2** For  $t > 0$ ,  $r \in [0, +\infty)$ ,  $\theta \in [-\pi, \pi]$ , the subelliptic heat kernel on  $\mathbb{H}^{2n+1}$  is given by

$$p_t^{\mathbb{H}^{2n+1}}(r, \theta) = \frac{\Gamma(n+1)e^{-n^2t + \frac{\pi^2}{4t}}}{(2\pi)^{n+2}t} \sum_{k \in \mathbb{Z}} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{e^{\frac{(y-i\theta-2k\pi i)^2 - u^2}{4t}} \sinh u \sin\left(\frac{\pi u}{2t}\right)}{(\cosh u + \cosh r \cosh y)^{n+1}} du dy. \quad (3.20)$$

*Proof.* It's easy to see that  $p_t^{\mathbb{H}^{2n+1}}$  satisfies the heat equation  $\frac{\partial}{\partial t} p_t^{\mathbb{H}^{2n+1}} = L p_t^{\mathbb{H}^{2n+1}}$ , where  $L$  is the sub-Laplacian defined on  $\mathbb{H}^{2n+1}$ . The initial condition is easily obtained by checking that

$$\sum_{k \in \mathbb{Z}} \int_{-\pi}^{\pi} \left( \frac{e^{-\frac{(\theta+2k\pi+iy)^2}{4t}}}{\sqrt{4\pi t}} \right) e^{i\lambda\theta} d\theta = \int_{-\infty}^{+\infty} \left( \frac{e^{-\frac{(\theta+iy)^2}{4t}}}{\sqrt{4\pi t}} \right) e^{i\lambda\theta} d\theta.$$

□

### 3.2 Asymptotics of the subelliptic heat kernel in small times

We mainly study the small-time asymptotics of the subelliptic heat kernel on  $\widetilde{\mathbb{H}^{2n+1}}$ . The ones on  $\mathbb{H}^{2n+1}$  are exactly the same, since by (3.2) we know that in small time the leading term in the sum is the term when  $k = 0$ . For this reason, we will use the notation  $p_t$  for both cases in the sequel.

Now to write the small-time asymptotics, let us first recall the other expression (3.14) of the heat kernel  $q_t$  we mentioned earlier. From it we easily obtain the following asymptotics of the heat kernel

$$q_t(\cosh \delta) = \frac{1}{(4\pi t)^{n+\frac{1}{2}}} \left( \frac{\delta}{\sinh \delta} \right)^n e^{-\frac{\delta^2}{4t}} \left( 1 + \left( n^2 - \frac{n(n-1)(\sinh \delta - \delta \cosh \delta)}{\delta^2 \sinh \delta} \right) t + O(t^2) \right), \quad (3.21)$$

where  $\delta \in [0, +\infty)$  is the Riemannian distance.

First, we study the asymptotics of the subelliptic heat kernel when  $t \rightarrow 0$  on the cut-locus of 0. Together with (3.17), let us first deduce the following small-time-asymptotics of the subelliptic heat kernel on the diagonal.

**Proposition 3.3** When  $t \rightarrow 0$ ,

$$p_t(0, 0) = \frac{1}{(4\pi t)^{n+1}} (A_n + B_n t + O(t^2)),$$

where  $A_n = \int_{-\infty}^{\infty} \frac{y^n}{(\sinh y)^n} dy$  and  $B_n = \int_{-\infty}^{\infty} \frac{y^n}{(\sinh y)^n} \left( n^2 - \frac{n(n-1)(\sinh y - y \cosh y)}{y^2 \sinh y} \right) dy$ .

*Proof.* We know that

$$p_t(0, 0) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{y^2}{4t}} q_t(\cosh y) dy.$$

Plug in (3.21), we have the desired small time asymptotics. □

**Proposition 3.4** For  $\theta \in \mathbb{R}$ ,  $t \rightarrow 0$ ,

$$p_t(0, \theta) = \frac{\theta^{n-1}}{2^{3n} t^{2n} (n-1)!} e^{-\frac{2\pi\theta + \theta^2}{4t}} (1 + O(t))$$

*Proof.* Let  $\theta \in \mathbb{R}$ , we have

$$p_t(0, \theta) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{(y-i\theta)^2}{4t}} q_t(\cosh y) dy.$$

Moreover, by (3.21), we know that

$$q_t(\cosh y) \sim_{t \rightarrow 0} \frac{1}{(4\pi t)^{n+\frac{1}{2}}} \left( \frac{y}{\sinh y} \right)^n e^{-\frac{y^2}{4t}}.$$

This gives

$$p_t(0, \theta) \sim_{t \rightarrow 0} \frac{e^{-\frac{\theta^2}{4t}}}{(4\pi t)^{n+1}} \int_{-\infty}^{\infty} \frac{y^n}{(\sinh y)^n} e^{-\frac{iy\theta}{2t}} dy.$$

By the residue theorem, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{y^n}{(\sinh y)^n} e^{-\frac{iy\theta}{2t}} dy &= -2\pi i \sum_{k \in \mathbb{Z}^+} \mathbf{Res} \left( \frac{e^{-\frac{iy\theta}{2t}} y^n}{(\sinh y)^n}, -k\pi i \right) \\ &= -2\pi i \sum_{k \in \mathbb{Z}^+} \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial y^{n-1}} \left[ \frac{e^{-\frac{iy\theta}{2t}} 2^n y^n (y + k\pi i)^n}{(e^y - e^{-y})^n} \right]_{y=-k\pi i}. \end{aligned}$$

Write  $W(y) = \frac{(y+k\pi i)^n}{(e^y - e^{-y})^n}$ ,  $W(y)$  is analytic around  $-k\pi i$ , and satisfies

$$W(-k\pi i) = \frac{1}{(-1)^{kn} 2^n}.$$

Hence the residue is

$$\frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial y^{n-1}} \left[ e^{-\frac{iy\theta}{2t}} 2^n y^n W(y) \right]_{y=-k\pi i}.$$

This is a product of  $e^{-\frac{iy\theta}{2t}}$  and a polynomial of degree  $n-1$  in  $1/t$ . We are only interested in the leading term which plays the dominant role when  $t \rightarrow 0$ . Thus we have the equivalence

$$\frac{-2\pi i}{(n-1)!} \frac{\partial^{n-1}}{\partial y^{n-1}} \left[ e^{-\frac{iy\theta}{2t}} 2^n y^n W(y) \right]_{y=-k\pi i} \sim_{t \rightarrow 0} \frac{(-1)^{kn+n} \pi^{n+1} \theta^{n-1}}{(n-1)! 2^{n-2} t^{n-1}} e^{-\frac{k\pi\theta}{2t}}$$

At the end, we conclude

$$p_t(0, \theta) \sim_{t \rightarrow 0} \frac{e^{-\frac{\theta^2}{4t}}}{(4\pi t)^{n+1}} \sum_{k \in \mathbb{Z}^+} \frac{(-1)^{kn+n} \pi^{n+1} \theta^{n-1}}{(n-1)! 2^{n-2} t^{n-1}} e^{-\frac{k\pi\theta}{2t}},$$

that is

$$p_t(0, \theta) = \frac{\theta^{n-1}}{2^{3n} t^{2n} (n-1)!} e^{-\frac{2\pi\theta + \theta^2}{4t}} (1 + O(t))$$

□

We now come to the points that do not lie on the cut-locus, i.e.,  $r \neq 0$ . First we deduce the case for  $(r, 0)$ .

**Proposition 3.5** *For  $r \in (0, +\infty)$ , we have*

$$p_t(r, 0) \sim \frac{e^{-\frac{r^2}{4t}}}{(4\pi t)^{n+\frac{1}{2}}} \left( \frac{r}{\sinh r} \right)^n \sqrt{\frac{1}{r \coth r - 1}}$$

as  $t \rightarrow 0$ .

*Proof.* By proposition 3.1,

$$p_t(r, 0) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{y^2}{4t}} q_t(\cosh r \cosh y) dy,$$

together with (3.21), it gives that

$$p_t(r, 0) \sim_{t \rightarrow 0} \frac{1}{(4\pi t)^{n+1}} \int_{-\infty}^{+\infty} e^{-\frac{(\cosh^{-1}(\cosh r \cosh y))^2 - y^2}{4t}} \left( \frac{\cosh^{-1}(\cosh r \cosh y)}{\sqrt{\cosh^2 r \cosh^2 y - 1}} \right)^n dy.$$

We can analyze it by the Laplace method. Notice that on  $\mathbb{R}$ , the function

$$f(y) = (\cosh^{-1}(\cosh r \cosh y))^2 - y^2$$

has a unique minimum at  $y = 0$ , where  $f(0) = r^2$  and

$$f''(0) = 2(r \coth r - 1).$$

Hence by the Laplace method, we can easily obtain the result. □

We can now extend the result to the case  $\theta \neq 0$  by applying the steepest descent method.

**Lemma 3.6** *For  $r \in (0, +\infty)$ ,  $\theta \in (-\infty, +\infty)$ ,*

$$f(y) = (\cosh^{-1}(\cosh r \cosh y))^2 - (y - i\theta)^2$$

*defined on the strip  $|\operatorname{Im}(y)| < \arccos\left(\frac{1}{\cosh r}\right)$  has a critical point at  $i\varphi(r, \theta)$ , where  $\varphi(r, \theta)$  is the unique solution in  $(-\arccos\left(\frac{1}{\cosh r}\right), \arccos\left(\frac{1}{\cosh r}\right))$  to the equation*

$$\varphi(r, \theta) - \theta = \cosh r \sin \varphi(r, \theta) \frac{\cosh^{-1}(\cosh r \cos \varphi(r, \theta))}{\sqrt{\cosh^2 r \cos^2 \varphi(r, \theta) - 1}}.$$



*Proof.* Let  $u = \cosh r \cos \varphi$ ,

$$\frac{\partial}{\partial \varphi} \left( \varphi - \cosh r \sin \varphi \frac{\cosh^{-1}(\cosh r \cos \varphi)}{\sqrt{\cosh^2 r \cos^2 \varphi - 1}} \right) = \frac{\sinh^2 r}{u(r, \theta)^2 - 1} \left( 1 - \frac{u(r, \theta) \cosh^{-1} u(r, \theta)}{\sqrt{u^2(r, \theta) - 1}} \right)$$

is negative, thus  $\theta \rightarrow \varphi - \cosh r \sin \varphi \frac{\cosh^{-1}(\cosh r \cos \varphi)}{\sqrt{\cosh^2 r \cos^2 \varphi - 1}}$  is a strictly decreasing on  $(-\infty, +\infty)$ , hence the uniqueness.  $\square$

Moreover, observe that at  $i\varphi(r, \theta)$ ,

$$f''(i\varphi(r, \theta)) = \frac{2 \sinh^2 r}{u(r, \theta)^2 - 1} \left( \frac{u(r, \theta) \cosh^{-1} u(r, \theta)}{\sqrt{u^2(r, \theta) - 1}} - 1 \right),$$

is a real positive number, where  $u(r, \theta) = \cosh r \cos \varphi(r, \theta)$ .

By using the steepest descent method we can deduce

**Proposition 3.7** *Let  $r \in (0, +\infty)$ ,  $\theta \in (-\infty, +\infty)$ . Then when  $t \rightarrow 0$ ,*

$$p_t(r, \theta) \sim_{t \rightarrow 0} \frac{1}{(4\pi t)^{n+\frac{1}{2}} \sinh r} \frac{(\cosh^{-1} u(r, \theta))^n}{\sqrt{\frac{u(r, \theta) \cosh^{-1} u(r, \theta)}{\sqrt{u^2(r, \theta) - 1}} - 1}} \frac{e^{-\frac{(\varphi(r, \theta) - \theta)^2 \tanh^2 r}{4t \sin^2(\varphi(r, \theta))}}}{(u(r, \theta)^2 - 1)^{\frac{n-1}{2}}},$$

where  $u(r, \theta) = \cosh r \cos \varphi(r, \theta)$ .

**Remark 3.8** *By symmetry, the sub-Riemannian distance from the north pole to any point on  $\widehat{\mathbb{H}^{2n+1}}$  only depends on  $r$  and  $\theta$ . If we denote it by  $d(r, \theta)$ , then from the previous propositions,*

(1) *For  $\theta \in \mathbb{R}$ ,*

$$d^2(0, \theta) = 2\pi|\theta| + \theta^2$$

(2) *For  $\theta \in \mathbb{R}$ ,  $r \in (0, +\infty)$ ,*

$$d^2(r, \theta) = \frac{(\varphi(r, \theta) - \theta)^2 \tanh^2 r}{\sin^2(\varphi(r, \theta))}$$

*The above formulas also work for  $\mathbb{H}^{2n+1}$  if we restrict  $\theta$  to  $[-\pi, \pi]$ .*

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